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On supersymmetric Lorentzian Einstein-Weyl spaces

Patrick Meessen[✉], Tomás Ortín[‡] and Alberto Palomo-Lozano[‡]

[✉] *HEP Theory Group, Departamento de Física, Universidad de Oviedo
Avda. Calvo Sotelo s/n, E-33007 Oviedo, Spain.*

[‡] *Instituto de Física Teórica UAM/CSIC, C/ Nicolás Cabrera 13-15
Ciudad Universitaria de Cantoblanco, E-28049 Madrid, Spain.*

In loving memory of Cte. Lozano Cid

ABSTRACT. We consider weighted parallel spinors in Lorentzian Weyl geometry in arbitrary dimensions, choosing the weight such that the integrability condition for the existence of such a spinor, implies the geometry to be Einstein-Weyl. We then use techniques developed for the classification of supersymmetric solutions to supergravity theories to characterise those Lorentzian EW geometries that allow for a weighted parallel spinor, calling the resulting geometries supersymmetric. The overall result is that they are either conformally related to ordinary geometries admitting parallel spinors (w.r.t. the Levi-Civita connection) or are conformally related to certain Kundt spacetime. A full characterisation is obtained for the 4 and 6 dimensional cases.

Over the last decades, spinorial fields parallelised by some (generalised) covariant derivative (we shall call such spinorial fields Killing spinors),¹ have become a prominent tool in physics as well as mathematics. In physics, such spinorial fields are usually related to supersymmetry and can be used to prove the positivity of the energy in physical systems, the stability of objects that preserve some residual supersymmetry or the non-renormalisability of the mass-charge relation for the so-called BPS objects which is of the utmost importance in, for example, String Theory's microscopic explanation of the entropy of supersymmetric black holes. In mathematics, one application which also appears frequently in the physics literature, is the link established by Hitchin between manifolds admitting parallel spinors and them having a special holonomy group [1], but can also be applied to more general settings, such as Weyl geometry [2, 3].²

Seeing the importance of such spinors it should not be surprising that in the last decade techniques were developed to extract the geometric information contained in the so-called *Killing spinor equations* (KSEs) *i.e.* the equations imposing the parallelity of the spinorial field under the generalised connection. The first systematic approach was made by Tod in ref. [5], taking leads from earlier work by Gibbons and Hull [6], who used the Newman-Penrose spinorial techniques [7] to obtain the supersymmetric solutions to a 4-dimensional supergravity theory usually referred to as minimal (or *pure*) $N = 2$ supergravity.³ In ref. [8], Gauntlett

¹ Observe that the concept of Killing spinor (equation) in the physics literature has a far broader meaning than in the mathematical literature.

² (Global) Spinors also have their applications and can *e.g.* be used to find generalised instanton equations [4].

³ In the supergravity literature it is customary to refer to specific theories by indicating the dimension of spacetime, d , and the number of minimal spinors, N , used to generate the supersymmetry transformations; the theory we just mentioned is therefore known as $d = 5$ $N = 2$ supergravity. However, in order not to give too many different meanings to d we will use the number n to mean dimensionality of spacetime, and will use the non-standard nomenclature “ $N = \#$ $n = \#$ supergravity”.

et al. overcame the inherent 4-dimensional restriction of the Newman-Penrose formalism by introducing the *spinor bilinear method* and classifying the supersymmetric solutions of 5-dimensional minimal $N = 1$ supergravity. This seminal article was the starting shot for a period of feverish activity in the supergravity literature, during which the supersymmetric solutions of the majority of supergravity theories were characterised, and even more powerful techniques, such as Gillard *et al.*'s spinorial geometry method [9], were developed.

The process of the spinor bilinear characterisation is basically split into two parts: first, given a rule for the parallel propagation of the spinor in terms of the relevant supergravity fields, one deduces the most general form of those fields compatible with the existence of a non-vanishing Killing spinor; the form of the fields thus obtained is called a supersymmetric field configuration. Seeing that the KSEs are linear in derivatives and the equations of motion (EOMs) are of second order, one cannot hope to obtain a recipe for solutions of the EOMs straight-away, and instead one uses the supersymmetric configurations as *Ansätze* to find (supersymmetric) solutions. In this sense, an observation made by Gauntlett *et al.* in ref. [8] (which was formalised in ref. [10]) reduces the amount of work necessary to find the conditions that a supersymmetric field configuration needs to fulfill, in order to give rise to a supersymmetric solution. The basic observation is that the fact that a solution preserves some supersymmetry means that there are relations between components of the equations of motion, meaning that there is a minimal set of independent components of the EOMs that, once satisfied, implies that all EOMs are satisfied. This observation is in fact completely general and depends only on a subset of the integrability conditions for the KSEs under consideration and on the spinorial structure used [10].

The first ones to realise that these techniques could be applied outside the realm of supersymmetry were the authors of ref. [11]. They considered a KSE similar to the one used in 5-dimensional minimal gauged supergravity, but with a De Sitter-like cosmological constant.⁴ As explained above, the integrability condition of their KSE places a constraint on the Ricci tensor corresponding to the Einstein's equations of motion, which then follow automatically from a solution to the KSE. The article goes on to classify the *timelike* solutions of the constructed theory, which turn out to show a four-dimensional hyper-Kähler torsion (HKT) base space dependence.

The work we present here follows similar lines, since we also consider a 'novel' KSE (in the sense that such KSE is not related *a priori* to any supersymmetric setting previously treated), and whose relevance becomes apparent once one analyses its integrability condition. Our motivation, however, is different from that of characterisations of solutions to (f)SUGRA theories. We are interested in classifying Lorentzian Einstein-Weyl spaces of arbitrary dimension, and the KSE is chosen in such a way that the integrability condition resembles the geometric constraint for a manifold to be Einstein-Weyl. The tools we will use for this work are the same ones as used in the programme of classification of solutions to supergravity theories, and we will split the problem at hand according to whether they employ a timelike or null vector field. The characterisation we give is of those EW spaces that arise from the existence of a *Killing spinor i.e.* a spinor that fulfills the KSE we propose, and it is in that sense that we refer to them as supersymmetric.

Section (1) introduces the spinorial rule, its integrability condition (which resembles the geometric constraint for Einstein-Weyl spaces) and a short manipulation on a vector bilinear

⁴This is in general incompatible with supersymmetry. Sometimes these theories are referred to as *fake* supergravities or (f)SUGRA.

valid for all dimensions and cases. Section (2) analyses all possible timelike cases, showing their triviality. Section (3) describes the *null* solutions for the $N = 1$, $n = 4$ case, while section (4) treats the $n = 6$ null case and section (5) the remaining cases. Section (6) recapitulates the work done. Three appendices are presented at the end for reference and completeness. Appendix (A) gives some basic knowledge (by no means exhaustive) of Weyl geometry and Einstein-Weyl spaces. Appendix (B) presents the spinorial notation we use in the article. Appendix (C) gives the geometrical description for Kundt waves, as they turn out to be relevant.

1 Covariant rule and the Einstein-Weyl condition

Consider the following rule for the covariant derivative of some spinor, which we shall take to be Dirac,

$$\nabla_a \epsilon = \frac{4-n}{4} A_a \epsilon + \frac{1}{2} \gamma_{ab} A^b \epsilon, \quad (1)$$

where n is the number of spacetime dimensions and A is just some real 1-form, which at this point is completely unconstrained. We will call the solutions ϵ of this equation Killing spinors and the corresponding metric and 1-form, a supersymmetric field configuration. Observe that with our choice of Dirac conjugate, the above rule implies

$$\nabla_a \bar{\epsilon} = \frac{4-n}{4} A_a \bar{\epsilon} - \frac{1}{2} A^b \bar{\epsilon} \gamma_{ab}. \quad (2)$$

A straightforward calculation of the integrability condition leads to

$$\frac{1}{2} \gamma_a \not{F} \epsilon = \frac{1}{2} W_{(ab)} \gamma^b \epsilon, \quad (3)$$

where $F \equiv dA$ is called the Faraday tensor and

$$W_{(ab)} = R(g)_{ab} - (n-2) \nabla_{(a} A_{b)} - (n-2) A_a A_b - g_{ab} [\nabla_c A^c - (n-2) A_c A^c], \quad (4)$$

which is readily identified with (the symmetric part of) the Ricci tensor in Weyl geometry (see appendix A for a small introduction).

Contracting the above integrability condition with γ^a one finds that

$$n \not{F} \epsilon = W \epsilon, \quad (5)$$

which when combined with eq. (3) leads to

$$\frac{1}{2} [W_{(ab)} - \frac{1}{n} W g_{ab}] \gamma^b \epsilon = 0. \quad (6)$$

In the Riemannian setting the above is enough to conclude that if we find a spinor ϵ satisfying eq. (1), then the underlying geometry is Einstein-Weyl. In the non-Riemannian setting this conclusion is not true: experience from the classification of supersymmetric solutions to supergravity theories shows instead that there are two quite different cases to be considered, namely the timelike or the null case. The sexer of these two cases is the norm of a particular vector-bilinear built out of the Killing spinor, which can be shown to be either zero or positive, hence the naming of the cases. The minimal set of equations of motion that need to be imposed in order to guarantee that all EOMs are satisfied, is different in each case: in the timelike case a supersymmetric field configuration automatically satisfies the EW condition,

whereas in the null case the minimal set consists of only one component of the EW condition, namely the one lying in the double direction of the null vector-bilinear.

Seeing the similarity of the integrability condition of the spinorial rule with the geometric constraint for EW spaces, it should not come as a surprise that eq. (1) is invariant under the following Weyl transformations

$$\begin{aligned} g &= e^{2w} \tilde{g} & , & & e^a &= e^w \tilde{e}^a , \\ A &= \tilde{A} + dw & , & & \theta_a &= e^{-w} \tilde{\theta}_a , \\ \epsilon &= e^{\alpha w} \tilde{\epsilon} & , & & \alpha &= \frac{4-n}{4} . \end{aligned} \quad (7)$$

This Weyl symmetry can in fact be used to obtain the r.h.s. of eq. (1), which would otherwise have to be wild-guessed: the Weyl connection, eq. (62), in the spinorial representation is given by $D_a = \nabla_a - \frac{1}{2} \gamma_{ab} A^b$, allowing us to rewrite eq. (1) as $D_a \epsilon = \frac{4-n}{4} A_a \epsilon$. In other words, we are dealing with a weighted Killing spinor in Weyl geometry.

The next step in the analysis is to define the bilinear $\hat{L} = L_\mu dx^\mu = \bar{\epsilon} \gamma_\mu \epsilon dx^\mu$, which (as shown in appendix B) is a real 1-form and, for a Lorentzian spacetime, is either timelike, $g(L, L) > 0$ in our conventions, or null, $g(L, L) = 0$. Independently of these details, however, we can always derive from the spinorial equation (1) the following differential rule for the bilinear

$$\nabla_a L_b = \frac{4-n}{2} A_a L_b - L_a A_b + \iota_L A g_{ab} , \quad (8)$$

whose totally antisymmetric part reads

$$d\hat{L} = \frac{6-n}{2} A \wedge \hat{L} , \quad (9)$$

singling out the $n = 6$ case as special, as \hat{L} is then closed.

We shall start the analysis by considering the timelike case.

2 Timelike solutions

Suppose that L is timelike and define $f \equiv g(L, L)$. We can straightforwardly use eq. (8) to find

$$df = (4-n) A f , \quad (10)$$

so that, as long as $n \neq 4$, the Weyl structure is exact and any supersymmetric EW-space is equivalent to a metrical space allowing for a parallel spinor. Bryant [12] has classified all the pseudo-Riemannian spaces admitting covariantly-constant spinors for a different number of dimensions. Then, this prescribes the *timelike* Einstein-Weyl metrics with Lorentzian signature in dimensions three (flat), five and six ($g = \mathfrak{R}^{1,n-5} \times \tilde{g}$, where \tilde{g} is a 4-dimensional Ricci-flat Kähler manifold). A general study for the remaining dimensions is still an open problem, as far as we know. However, Galaev & Leistner [13] provide a partial answer by giving a blueprint for the geometry of simply-connected, complete Lorentzian spin manifolds that admit a Killing spinor (see theorem 1.3 therein).

For the $n = 4$ case, we use the same building blocks as in ref. [14] to set up the whole calculus of spinor bilinears. We deal with the spinor structure of $N = 2$ $n = 4$ supersymmetry, which allows us to decompose a Dirac spinor in $n = 4$ as a sum of two Majorana spinors, which we can project onto the anti-chiral part, denoted ϵ_I ($I = 1, 2$), and the chiral part, denoted by ϵ^I . Here the position of the I -index indicates exclusively the chirality, and the

chiralities are interchanged by complex conjugation *i.e.* $(\epsilon_I)^* = \epsilon^I$, so the theory has two independent spinors. Doing this decomposition, the rule eq. (1) can be written as

$$\nabla_a \epsilon_I = \frac{1}{2} \gamma_{ab} A^b \epsilon_I \quad \text{and} \quad \nabla_a \epsilon^I = \frac{1}{2} \gamma_{ab} A^b \epsilon^I. \quad (11)$$

Using the spinors one can then construct (see ref. [14]) a complex scalar $X \equiv \frac{1}{2} \epsilon^{IJ} \bar{\epsilon}_I \epsilon_J$, 3 complex 2-forms Φ^x ($x = 1, 2, 3$) that will not play any rôle in what follows, and 4 real 1-forms $V^a = i \bar{\epsilon}^I \gamma^a \epsilon_I$. These 4 1-forms form a linearly independent base and can be used to write the metric, g , as

$$4|X|^2 g = \eta_{ab} V^a \otimes V^b, \quad (12)$$

whence $V^0 \sim L$. Given the definitions of the bilinears we can calculate

$$dX = 0, \quad (13)$$

$$dV^a = A \wedge V^a, \quad (14)$$

meaning that X is just a complex constant. The integrability condition of eq. (14) is $F \wedge V^a = 0$ which, due to the linear-independency of the V^a implies that $F = 0$. Locally, then, we can transform A to zero and introduce coordinates x^a such that $V^a = 4|X|^2 dx^a$, resulting in a Minkowski metric. Whence, in $n = 4$ a *timelike* supersymmetric Lorentzian EW space is locally conformal to Minkowski space.

The conclusion then w.r.t. the *timelike* solutions to the rule (1) is that they are trivial in the sense that they are always related by a Weyl transformation to a Lorentzian space admitting Killing spinors, *i.e.* spinors satisfying the rule $\nabla_a \epsilon = 0$.

The analysis of the null cases is more involved, mainly due to a lack of systematics in the bilinears, the exception being the vector bilinear L as one can see from eq. (8), but also because the bilinear approach to classification of supersymmetric solutions becomes unwieldy for $n > 6$. In stead of attempting to do a complete analysis in all the cases where the bilinear approach can be applied, we shall analyse the cases $n = 4$ and $n = 6$ explicitly, and then give some generic comments in section (5).

3 Null $N = 1$ $n = 4$ solutions

The natural starting point, seeing the explicit case treated in the foregoing section, would be the null case in $n = 4$ $N = 2$. Prior experience with this case in supergravity, however, shows that this case is related to the simpler case of $n = 4$ $N = 1$ supergravity [15], a theory for which the vector bilinear L is automatically a null vector. In $n = 4$ $N = 1$ sugra the spinor is a Weyl spinor, and one can see that the KSE (1) is compatible with the truncation of ϵ to a chiral spinor, and in this section we shall henceforth take ϵ to be a Weyl spinor.

The first rule we can derive for the bilinear is

$$\nabla_a L_b = -L_a A_b + \iota_L A g_{ab}, \quad (15)$$

which is already enough to see that L^\flat is a geodesic null vector. The antisymmetric and symmetric parts of the above equation read

$$d\hat{L} = A \wedge \hat{L}, \quad (16)$$

$$\nabla_{(a} L_{b)} = -A_{(a} L_{b)} + \frac{1}{3} \nabla \cdot L g_{ab}. \quad (17)$$

There is another bilinear that can be constructed [15], which is a 2-form defined as $\Phi_{ab} = \bar{\epsilon}\gamma_{ab}\epsilon$ and using the propagation rule we can deduce

$$\nabla_a \Phi_{bc} = 2\Phi_{a[b}A_{c]} - 2g_{a[b}\Phi_{c]}dA^d, \quad (18)$$

which through antisymmetrisation gives rise to

$$d\Phi = 2A \wedge \Phi. \quad (19)$$

Eq. (16) implies that $\hat{L} \wedge d\hat{L} = 0$, whence \hat{L} is hypersurface orthogonal, and we can use the Frobenius theorem to introduce two real functions u and P such that $\hat{L} = e^P du$. Since by eq. (16) above \hat{L} has gauge charge 1 under A , we can perform a Weyl-gauge transformation to take $P = 0$, as to obtain $\hat{L} = du$. This further implies that $A = \Upsilon \hat{L}$, where Υ is a real function whose coordinate dependence needs to be deduced, and also $\iota_L A = 0$. Furthermore, we see that $d^\dagger \hat{L} = 0$ and $\nabla_L L = 0$, *i.e.* L is the tangent vector to an affinely parametrised null geodesic.

Observe that we can apply the same reasoning for eq. (9) in dimensions different from six: as long as $n \neq 6$ we can always use a Weyl transformation as to fix $\hat{L} = du$ and write $A = \Upsilon \hat{L}$. The fact that in the case $n = 6$ the 1-form \hat{L} is automatically closed has profound implications, as will be shown in section (4).

Having fixed the Weyl symmetry, we can introduce a normalised null tetrad [7] and a corresponding coordinate representation by

$$\begin{aligned} \hat{L} &= du & , & \quad L = \partial_v , \\ \hat{N} &= dv + Hdu + \varpi dz + \bar{\varpi} d\bar{z} & , & \quad N = \partial_u - H\partial_v , \\ \hat{M} &= Udz & , & \quad M = -\bar{U}^{-1}(\partial_{\bar{z}} - \bar{\varpi}\partial_v) , \\ \hat{\bar{M}} &= \bar{U}d\bar{z} & , & \quad \bar{M} = -U^{-1}(\partial_z - \varpi\partial_v) , \end{aligned} \quad (20)$$

for which the metric reads

$$\begin{aligned} g &= \hat{L} \otimes \hat{N} + \hat{N} \otimes \hat{L} - \hat{M} \otimes \hat{\bar{M}} - \hat{\bar{M}} \otimes \hat{M} \quad \longrightarrow \\ ds^2 &= 2du(dv + Hdu + \varpi dz + \bar{\varpi} d\bar{z}) - 2|U|^2 dz d\bar{z} . \end{aligned} \quad (21)$$

A straightforward calculation shows that the constraint (17) implies that

$$\Upsilon = -\partial_v H \quad , \quad \partial_v \varpi = 0 \quad , \quad \partial_v \bar{\varpi} = 0 \quad , \quad \partial_v |U|^2 = 0 , \quad (22)$$

so that the only v -dependence resides in the function H , and we determined the gauge field A in terms of H .

In $N = 1$ $n = 4$ one can see that $\Phi = \hat{L} \wedge \hat{\bar{M}}$ (see *e.g.* [14, eq. (70)]). Combining this with eq. (19) we see that

$$0 = \hat{L} \wedge d\hat{\bar{M}} = d\bar{U} \wedge d\bar{z} \wedge du \quad \text{whence} \quad \bar{U} = \bar{U}(u, \bar{z}) . \quad (23)$$

This result means that we can take $U = 1$ by a suitable coordinate transformation $Z = Z(u, z)$ such that $\partial_z Z = U$, which leaves the chosen form of the metric invariant.

In order to finish the analysis, let us investigate eq. (18). As $A \sim \hat{L}$ we have that $\iota_A \Phi \sim \iota_L \Phi = 0$ and we find that $\nabla_a \Phi_{bc} = 2\Upsilon \Phi_{a[b}L_{c]}$. Combining this with $\Phi_{ab} = 2L_{[a}\bar{M}_{b]}$ we find that

$$0 = L_{[b} \nabla_a \bar{M}_{c]} , \quad (24)$$

which can be evaluated on the chosen coordinate basis to give

$$0 = \partial_{\bar{z}}\varpi - \partial_z\bar{\varpi} \quad \text{which implies: } \varpi = \partial_z B, \bar{\varpi} = \partial_{\bar{z}} B, \quad (25)$$

where B is a real function. As is well-known, one can then get rid of ϖ altogether by a suitable shift of the coordinate $v \rightarrow v - B$.

The end result of this analysis is that, given the fact that the spinor ϵ is taken to be a Weyl spinor, any solution⁵ to the equation (1) is related by a Weyl transformation to

$$ds_{(4)}^2 = 2du(dv + Hdu) - 2dzd\bar{z}, \quad (26)$$

$$A = -\partial_v H du, \quad (27)$$

Actually, this metric is a special case of a more-general metric, referred to as a Kundt metric in the physics literature (see appendix (C) for more information), a type of metric that appears naturally in the null case of not only supergravity [18] solutions, but also fake supergravity solutions, see *e.g.* refs. [14, 19] and [20].

At this point we would like to recall what was mentioned in section (1) above about pseudo-Riemannian signatures and certain EOMs (the EW conditions in this case) not having to be explicitly checked. Since we are trying to give a prescription for EW spaces, we obviously need to satisfy eq. (66). An explicit calculation shows that the integrability conditions (6) are automatically satisfied, with the only non-trivial component being $W(N, N) L_c \gamma^c \epsilon$. Adapting the Fierz identities to the null case scenario, one obtains the constraint $L_c \gamma^c \epsilon = 0$ (see *e.g.* eq. (5.12) of ref. [16]), satisfying this way the integrability condition, and hence we see that we have a solution to the KSE.

However, we still need to ensure that the local geometry (26) indeed solves all EW conditions (66), and we must therefore impose by hand that $W(N, N) = 0$. A small calculation shows that this implies that H must satisfy the following differential equation

$$\partial_u \partial_v H - H \partial_v^2 H = \partial \bar{\partial} H. \quad (28)$$

We can find a four-dimensional generalisation of the Weyl-scalar-flat EW geometry obtained by Calderbank & Dunajski in [17] by using a function H of the form

$$H = v\partial F + v\bar{\partial}\bar{F} + \bar{z}\partial_u F + z\partial_u \bar{F} \quad \text{where } F = F(u, z). \quad (29)$$

It gives rise to a non-trivial EW space as long as $\partial^2 F \neq 0$.

4 Null $N = (1, 0)$ $n = 6$ solutions

As in the foregoing section we will consider the spinor ϵ to be chiral which not only implies that the vector bilinear is null, but also that we can use the results of Gutowski *et al.* [21], who classified the supersymmetric solutions of ungauged chiral supergravity in 6 dimensions, *i.e.* minimal $n = 6$ $N = (1, 0)$ supergravity. This theory is in itself quite curious, and so are the spinor bilinears: there is only a null vector L and a triplet of selfdual 3-forms $\Phi_{(3)}^r$ ($r = 1, 2, 3$). These bilinears are defined by

$$\begin{aligned} L_a &\equiv -\varepsilon^{IJ} \epsilon_I^c \gamma_a \epsilon_J, & \epsilon_I^c \gamma_a \epsilon_J &= -\frac{1}{2} \varepsilon_{IJ} L_a, \\ \Phi_{abc}^r &\equiv i [\sigma^r]^{IJ} \epsilon_I^c \gamma_{abc} \epsilon_J, & \epsilon_I^c \gamma_{abc} \epsilon_J &= \frac{i}{2} [\sigma^r]_{IJ} \Phi_{abc}^r, \end{aligned} \quad (30)$$

⁵By solution we refer to a geometry that arises from the existence of a spinor that fulfills eq. (1).

where $\epsilon^c = \epsilon^T \mathcal{C}$ means the Majorana conjugate.

These bilinears satisfy the following Fierz-relations

$$L_a L^a = 0, \quad (31)$$

$$\iota_L \Phi_{(3)}^r = 0 \quad \longrightarrow \quad \hat{L} \wedge \Phi_{(3)}^r = 0, \quad (32)$$

$$\Phi^r{}^{fab} \Phi_{fcd}^s = 4\delta^{rs} L^{[a} L_{[c} \eta_{d]}^{b]} - \varepsilon^{rst} L^{[a} \Phi^{t|b]}{}_{cd} + \varepsilon^{rst} L_{[c} \Phi^{t|ab}{}_{d]}. \quad (33)$$

Seeing eqs. (32) and (33) we find that $\Phi_{(3)}^r = \hat{L} \wedge K_{(2)}^r$ with $\iota_L K_{(2)}^r = 0$.

Using the definitions of the bilinears we can use the rule eq. (1) to calculate the effect of parallel-transporting them. The results is that for an arbitrary vector field X we have

$$\nabla_X \hat{L} = -\iota_X A \hat{L} - \iota_X \hat{L} A + \iota_L A \hat{X}, \quad (34)$$

$$\nabla_X \Phi^r = -\iota_X A \Phi^r + \hat{X} \wedge \iota_{A^\flat} \Phi^r - A \wedge \iota_X \Phi^r, \quad (35)$$

From eq. (34) it is clear that L is a null geodesic, *i.e.* $\nabla_L L = 0$, and, as we already knew from (9), $d\hat{L} = 0$.

At this point then we can, as before, introduce a Vielbein adapted to the null nature of L in terms of the natural coordinates v , u and y^m ($m = 1, \dots, 4$) as

$$\begin{aligned} E^+ &= du & \theta_+ &= \partial_u - H\partial_v, \\ E^- &= dv + Hdu + S_m dy^m & \theta_- &= \partial_v, \\ E^i &= e_m{}^i dy^m & \theta_i &= e_i{}^m [\partial_m - S_m \partial_v], \end{aligned} \quad (36)$$

where $\hat{L} \equiv E^+$ and $L \equiv \theta_-$. As usual we can then define the metric on the base space by $h_{mn} \equiv e_m{}^i e_n{}^i$ and we can write the full 6-dimensional Kundt metric as

$$ds_{(6)}^2 = 2du \left(dv + Hdu + \hat{S} \right) - h_{mn} dy^m dy^n. \quad (37)$$

We can expand the 2-forms as $2K^r \equiv K_{ij}^r E^i \wedge E^j$ w.r.t. the above Vielbein, and by choosing the light-cone directions such that $\varepsilon^{+-1234} = 1 = \varepsilon^{1234}$, we see that $\star_{(4)} K^r = -K^r$. Defining the (1,1)-tensors J^r by means of $h(J^r X, Y) \equiv K^r(X, Y)$, we can see that eq. (33) implies

$$J^r J^s = -\delta^{rs} + \varepsilon^{rst} J^t, \quad (38)$$

so that the 4-dimensional base space is always going to be an *almost quaternionic manifold*.

At this point we will fix part of the Weyl gauge symmetry by imposing the gauge-fixing condition $\iota_L A = 0$ and consequently we can expand the gauge field as

$$A = \Upsilon \hat{L} + A_m dy^m. \quad (39)$$

Using this expansion and the explicit form of the Vielbein in terms of the coordinates, we can analyse eq. (34), resulting in

$$\Upsilon = -\frac{1}{2} \partial_v H, \quad (40)$$

$$\partial_v \hat{S} = -2A, \quad (41)$$

$$0 = \partial_v h_{mn}. \quad (42)$$

Contrary to what is usually the case in (fake) supergravities, we do not know the full v -dependence of H and therefore we cannot completely fix the v -dependence of the unknowns.

The above results comprise all the information contained in eq. (34).

In order to analyse the content of eq. (35) we first take $X = L$ to find that $\nabla_L \Phi^r = 0$, which when evaluated in the chosen coordinate system implies $\partial_v K_{mn}^r = 0$. This innocuous result fixes, however, the v -dependence of A : from the totally antisymmetric part of eq. (35) one obtains

$$d\Phi^r = 2A \wedge \Phi^r \longrightarrow 0 = \hat{L} \wedge [dK^r - 2A \wedge K^r] , \quad (43)$$

where we introduced the exterior derivative on the base space $d \equiv dy^m \partial_m$. As the K s are v -independent and $\hat{L} = du$, we see that the consistency of the above equation requires A to be v -independent. Then, we also obtain from eq. (41) that

$$\hat{S} = -2v A + \varpi \quad (\partial_v \varpi_m = 0) . \quad (44)$$

It should be clear from eq. (43) that the y -dependence of the K s is given by the equation

$$dK^r = 2A \wedge K^r \quad \text{whose integrability condition reads} \quad F \wedge K^r = 0 , \quad (45)$$

where we defined $F = dA$. Actually, the last equation implies, as one can easily verify, that F is selfdual, *i.e.* $\star_{(4)} F = F$, whence A is a selfdual connection or in physics-speak an \mathbb{R} -instanton.

The analysis of eq. (35) in the direction $X = \theta_+$ is straightforward and leads to the following constraints on the spin connection

$$\omega_{+-k} K_{kj}^r = -A_k K_{kj}^r , \quad (46)$$

$$0 = \omega_{+i}{}^k K_{kj}^r + \omega_{+j}{}^k K_{ik}^r . \quad (47)$$

By using the results in appendix C, we see that eq. (46) is automatically satisfied. A small investigation in eq. (47) shows that it implies the base space 2-form $\omega_{+ij} E^i \wedge E^j$ to be selfdual! Coupling this observation with eq. (78) and taking into account F 's selfduality, we see that the base space 2-form $2\Omega = \Omega_{ij} E^i \wedge E^j$, whose components are defined by

$$\Omega_{ij} \equiv 2D_{[i} \varpi_{j]} + 2e_{[i}{}^m \partial_u e_{j]m} \quad (\text{where: } D\varpi \equiv d\varpi - 2A \wedge \varpi) , \quad (48)$$

has to be selfdual, *i.e.* $\star_{(4)} \Omega = \Omega$.

In order to completely drain eq. (35) of information we need to consider X lying on the base space. Let X be such a vector. Then, we find that

$$\nabla_X^{(\lambda)} K^r = X^\sharp \wedge \star_{(4)} [A \wedge K^r] - A \wedge \iota_X K^r , \quad (49)$$

where $\nabla^{(\lambda)}$ is the ordinary spin connection on the base space using the λ s in eq. (78). Following ref. [11] we can then introduce a torsionful connection $\bar{\nabla}_X Y \equiv \nabla_X^{(\lambda)} Y - S_X Y$ with the torsion being totally antisymmetric and proportional to the Hodge dual of the \mathbb{R} -gauge field, *i.e.*

$$h(S_X Y, Z) \equiv -[\star_{(4)} A](X, Y, Z) , \quad (50)$$

such that eq. (49) can be written compactly as $\bar{\nabla} K^r = 0$. Almost quaternionic manifolds admitting a torsionful connection parallelising the almost quaternionic structure are called *Hyper-Kähler Torsion manifolds*, HKT manifolds for short, a name that first appeared in [22] to describe the geometry of supersymmetric sigma-model manifolds with torsion [23].

As pointed out in ref. [11], we can make use of the residual Weyl symmetry in eq. (7) with $w = w(y)$, *i.e.* a Weyl transformation depending only on the coordinates of the base space, to gauge-fix the condition $\mathbf{d}^\dagger \mathbf{A} = 0$. This immediately implies that the torsion \mathbf{S} is closed, and the resulting mathematical 4-dimensional structure is called a *closed HKT manifold*. Let us mention, even though it will not be needed, that the coordinate transformation $v \rightarrow v + \Lambda(y)$, induces the 'gauge' transformation $\varpi \rightarrow \varpi + \mathbf{D}\Lambda$.

Thus far, the analysis has shown that the pair (g, A) admits a solution to eq. (1) iff g is the metric of a Kundt wave whose base space is a u -dependent family of HKT-spaces. Given such a family of HKT spaces we can find the 1-form ϖ by imposing selfduality of the 2-form Ω in eq. (48) and then the only indeterminate element of the metric is the wave profile H . This analysis has given us the necessary conditions for the existence of a non-null spinor satisfying eq. (1). It remains to be checked that they are also sufficient by direct substitution into eq. (1).

A quick calculation of the $(-)$ component, leads to $\theta_- \epsilon = 0$, whence the spinor is v -independent. The $(+)$ -component leads, after using the constraint $\gamma^+ \epsilon = 0$, to

$$\partial_u \epsilon = -\frac{1}{4} T_{ij} \gamma^{ij} \epsilon = 0, \quad (51)$$

where the last step follows from the selfduality of T (see eq. (79)) and the chirality of the spinor ϵ . We conclude that the spinor is also u -independent. Giving the i components of eq. (1) a similar treatment we end up with

$$\nabla_i^{(\lambda)} \epsilon = \frac{1}{2} \tilde{\gamma}_{ij} A_j \epsilon, \quad (52)$$

where we have defined $\tilde{\gamma}^i \equiv i\gamma^i$, so $\{\tilde{\gamma}^i, \tilde{\gamma}^j\} = 2\delta^{ij}$, in order to obtain a purely Riemannian spinorial equation.

As one can readily see from eq. (1), the above equation is nothing more than its Riemannian version for four-dimensional spaces: this kind of spinorial equations was studied by Moroianu in ref. [2] who investigated Riemannian Weyl geometries admitting spinor fields parallel w.r.t. the Weyl connection. For $n \neq 4$ he found that any such Weyl structure was closed, whereas in $n = 4$ he found the HKT structure outlined above. Furthermore, he showed that, if the 4-dimensional space is compact, then the HKT structure is conformally related to either a flat torus, a K3 manifold or the Hopf surface $S^1 \times S^3$ with the standard, locally flat metric (see *e.g.* [24]).

The integrability condition of eq. (52) implies that the Ricci tensor of the metric \mathbf{h} has to satisfy

$$\mathbf{R}(\mathbf{h})_{ij} = 2\nabla_{(i}^{(\lambda)} A_{j)} + 2A_i A_j + h_{ij} \left(\nabla_i^{(\lambda)} A_i - 2A^2 \right), \quad (53)$$

which, by comparison with eq. (4), is equivalent to saying that the pair (\mathbf{h}, A) forms a Ricci-flat Weyl geometry *i.e.* $\mathbf{W}_{(ij)} = 0$.

As we did in section (3), we impose the Einstein-Weyl equations in those directions in which it is not trivially satisfied, *i.e.* in the $(++)$ -direction. This, in turn, fixes the function H , which was otherwise unknown. At this point, however, we would like to impose the simplifying restriction that the HKT structure on the base space does not depend on u . The motivation for this simplifying adjustment has to do with the difficulty of finding analytic

solutions to the differential equation resulting from a u -dependent base space. A calculation of the $(++)$ -components of the E-W equations then shows that

$$2\theta_+\theta_-H + (\theta_-H)^2 = \left(\nabla_i^{(\lambda)} - S_i\theta_- - 4A_i \right) (\partial_i - S_i\theta_- - 2A_i) H, \quad (54)$$

where we have allow for a u -dependence of H .

To summarise, any solution to the $N = (1, 0)$ $n = 6$ null scenario is once again prescribed by a Kundt wave of the form eq. (37) constrained by eqs. (44), (48) and (54), whose 4-dimensional base space is given by a v -independent, closed HKT manifold subject to eqs. (53), and the gauge connection being that of an \mathbb{R} -instanton.

5 Remaining null cases

Having treated the null cases in $n = 4$ and $n = 6$, we are ready to make some general comments on the null case in other dimensions. First of all, as was pointed out in section (3), as long as $n \neq 6$ we can use a Weyl transformation to introduce a coordinate u such that $\hat{L} = du$ and then also $A = \Upsilon \hat{L}$. Choosing the coordinate v to be aligned with the flow of $L (= \partial_v)$, we can introduce the base space coordinates y^m ($m = 1, \dots, n-2$) and a Vielbein similar to the one in eq. (74), so that the metric is always of the form

$$ds_{(n)}^2 = 2du(dv + Hdu + S_m dy^m) - h_{mn} dy^m dy^n, \quad (55)$$

where $h_{mn} \equiv e_m^i e_n^i$. This is again a Kundt metric, and evaluating the symmetric part of eq. (8) in this coordinate system, we get the following restrictions

$$\Upsilon = -\frac{2}{n-2} \partial_v H, \quad \partial_v S_m = 0, \quad \partial_v h_{mn} = 0, \quad (56)$$

so that the whole v -dependence resides in H and Υ only. Following the convention in section (4), we shall call the v -independent part of \hat{S} by ϖ , so that in the $n \neq 6$ case we have $\hat{S} = \varpi$.

With this information, and the constraint of u -independence imposed, we can proceed to analyse the spinorial rule. The KSE in the v -direction is automatically satisfied (*i.e.* $\partial_v \epsilon = 0$) and the remaining directions are

$$0 = \nabla_i^{(\lambda)} \epsilon, \quad (57)$$

$$\partial_u \epsilon = \frac{1}{8} [d\varpi]_{ij} \gamma^{ij} \epsilon. \quad (58)$$

Eq. (57) clearly states that the base space must be a Riemannian manifold of special holonomy. The integrability condition of the above two equations then is that

$$0 = \left[\nabla_i^{(\lambda)} (d\varpi)_{kl} \right] \gamma^{kl} \epsilon \quad \text{which implies} \quad \left[d^\dagger d\varpi \right]_i \gamma^i \epsilon = 0, \quad (59)$$

so that $d^\dagger d\varpi = 0$.⁶ Using the coordinate transformation $v \rightarrow v + \Lambda(y)$ we can always take $d^\dagger \varpi = 0$, whence $\varpi \in \text{Harm}^1(\mathcal{B})$, *i.e.* ϖ is a harmonic 1-form on the base space.⁷

⁶The same constraint can be obtained through explicit evaluation of the Einstein-Weyl equations.

⁷Bochner's theorem states that any harmonic 1-form on a compact, oriented Ricci-flat manifold is parallel, which implies that in that case the Killing spinor is u -independent. In the non-compact case, however, there is no such theorem as can be envisaged by taking the base space to be \mathbb{R}^{n-2} and to take $2\varpi \equiv f_{mn} x^m dx^n$, where the f_{nm} 's are constants.

Given this input, the condition for such a pair (g, A) to be an Einstein-Weyl manifold is

$$2\partial_u\partial_v H - 2H\partial_v^2 H + 2\frac{n-4}{n-2}(\partial_v H)^2 = -(\nabla^{(\lambda)} - \varpi)^i \theta_i H. \quad (60)$$

The factor on the r.h.s. of the above equations becomes, in the $\varpi = 0$ limit, the d'Alembertian on the base space, and we make contact with eq. (28). This shows that the $n = 4$ case is a subcase of the general one studied in this section, where one was allowed to use the 2-form Φ to get rid of \hat{S} . $n = 6$, however, is an independent case where the characteristic behaviour of the theory in that dimension (see *e.g.* eq. (9)) nurtures the HKT structure.

6 Summary and conclusions

In this work we have presented a characterisation of supersymmetric Einstein-Weyl spaces with Lorentzian signature in n arbitrary dimensions. We have done this by making use of the techniques developed for the classification of supergravity solutions. In particular, we assumed the existence of a spinor ϵ satisfying eq. (1). It is in this sense that our solutions have a supersymmetric character. We then proceeded to build and analyse the bilinears that can be constructed from ϵ , which shape the resulting geometry.

We have found that (for most dimensions) those spaces arising from a vector bilinear which is timelike are trivial, in the sense that they are conformally related to a space admitting a Killing spinor. The odd duck in the pond is the 4-dimensional case, for which the only *timelike* solution actually turns out to be Minkowski space, which coincides with which was already known for parallel spinors [12]. The null case solutions are given by a Kundt metric and a prescribed Weyl gauge field. It is worth mentioning that the special structure of the $n = 6$ case determines that the base space is given by a closed *Hyper-Kähler Torsion* manifold.

As a closing paragraph let us consider the case $n = 3$: in that case one can see that eq. (60), once one takes into account the fact that one perform coordinate transformations such that $h = 1$ and $\varpi = 0$, corresponds to the dispersionless Kadomtsev-Petviashvili equation. As shown in ref. [25, sec. 10.3.1.3], the thus obtained class of 3-dimensional EW spaces is the unique class of 3-dimensional EW spaces of Lorentzian signature admitting a weighted covariantly constant null vector. Furthermore, the supersymmetric class can be obtained by the Jones-Tod construction on a conformal space of neutral signature admitting an anti-selfdual Null-Kähler structure [25], a geometric structure which admits a parallel spinor. Evidently, there are n -dimensional EW spaces, as there are 3-dimensional examples, that are not supersymmetric, and it would be interesting to get a better handle on them.

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A A short introduction to Einstein-Weyl geometry

A Weyl manifold is a manifold \mathcal{M} of dimension n , a conformal class $[g]$ of metrics on \mathcal{M} , and a torsionless connection \mathbb{D} , which preserves the conformal class, *i.e.*

$$\mathbb{D} g = 2A \otimes g , \quad (61)$$

for a chosen reference $g \in [g]$ and $A \in \Omega(\mathcal{M})$. Using the above definition, we can express the connection $\mathbb{D}_X Y$ as

$$\mathbb{D}_\mu Y_\nu = \nabla_\mu^g Y_\nu + \gamma_{\mu\nu}{}^\rho Y_\rho \quad \text{with} \quad \gamma_{\mu\nu}{}^\rho = g_\mu{}^\rho A_\nu + g_\nu{}^\rho A_\mu - g_{\mu\nu} A^\rho , \quad (62)$$

where ∇^g is the Levi-Civita connection for the chosen $g \in [g]$. We define the curvature of this connection as usual, *i.e.* $[\mathbb{D}_\mu, \mathbb{D}_\nu] Y_\rho = -W_{\mu\nu\rho}{}^\sigma Y_\sigma$, and which we can use to define the associated Ricci curvature as $W_{\mu\nu} \equiv W_{\mu\sigma\nu}{}^\sigma$. A calculation shows that the Ricci tensor is not symmetric, which was to be suspected as we have a connection with non-vanishing contorsion, and we have

$$W_{[\mu\nu]} = -\frac{n}{2} F_{\mu\nu} \quad \text{other notations: } F = dA = \rho^\mathbb{D} , \quad (63)$$

$$W_{(\mu\nu)} = R(g)_{\mu\nu} - (n-2)\nabla_{(\mu} A_{\nu)} - (n-2) A_\mu A_\nu - g_{\mu\nu} [\nabla_\sigma A^\sigma - (n-2) A_\sigma A^\sigma] \quad (64)$$

The Ricci-scalar is then of course defined as $W \equiv W_\sigma{}^\sigma$, which explicitly reads

$$W = R(g) - 2(n-1) \nabla_\sigma A^\sigma + (n-1)(n-2) A_\sigma A^\sigma . \quad (65)$$

The 1-form A acts as gauge field gauging an \mathbb{R} -symmetry, and this is also the reason why we have been talking about a conformal class of metrics on \mathcal{M} . In fact, under a transformation $g_{\mu\nu} \rightarrow e^{2w} g_{\mu\nu}$, we have that $A \rightarrow A + dw$ and $W \rightarrow e^{-2w} W$, whereas $W_{\mu\nu\rho}{}^\sigma$ and $W_{\mu\nu}$ are conformally invariant. In this sense, we say that an EW structure is trivial if the field strength $F = dA = 0$, *i.e.* locally the Weyl connection is conformally vanishing.

A Weyl manifold is said to be Einstein-Weyl if the curvatures satisfy

$$W_{(\mu\nu)} = \frac{1}{n} g_{\mu\nu} W . \quad (66)$$

A metric g in the conformal class is said to be *standard* or *Gauduchon* if it is such that

$$d \star A = 0 \quad \text{or equivalently} \quad \nabla_\sigma A^\sigma = 0 , \quad (67)$$

where the \star is taken w.r.t. the chosen metric. Gauduchon [26] showed that on a compact EW manifold there always exists a standard metric, and Tod [27] then went on to show that on compact EW manifolds this implies that A^\flat is a Killing vector of the metric, *i.e.* it generates an isometry of g .

B Spinors in $\text{SO}(1, d-1)$

On $\mathbb{R}^{1,n-1}$ we shall put the mostly negative metric $\eta = \text{diag}(+, [-]^{n-1})$ and take the γ -matrices to satisfy

$$\{ \gamma_a, \gamma_b \} = 2\eta_{ab} . \quad (68)$$

We use a unitary representation of the γ -matrices, which implies that $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$. Choosing the Dirac conjugation matrix $\mathcal{D} = \gamma_0$, we define the Dirac conjugate of a spinor ψ by $\bar{\psi} \equiv \psi^\dagger \mathcal{D}$ and find that

$$\mathcal{D}\gamma_a\mathcal{D}^{-1} = \gamma_a^\dagger \quad \text{and} \quad \mathcal{D}\gamma_{ab}\mathcal{D}^{-1} = -\gamma_{ab}^\dagger \quad (69)$$

Defining the 1-form $L = L_a e^a$ by means of $L_a \equiv \bar{\psi}\gamma_a\psi$ which is then automatically real:

$$L_a^* = \overline{\bar{\psi}\gamma_a\psi} = \psi^T (\mathcal{D}\gamma_a)^* \psi^* = \psi^\dagger (\mathcal{D}\gamma_a)^\dagger \psi = \bar{\psi}\mathcal{D}^{-1}\gamma_a^\dagger\mathcal{D}\psi = \bar{\psi}\gamma_a\psi = L_a, \quad (70)$$

where a perhaps expected -1 sign in the third step is absent as we are dealing with classical (commuting) spinors.

In terms of the components we have that $L_a = \epsilon^\dagger \mathcal{D}\gamma_a \epsilon$ and it is clear that $L_0 = \epsilon^\dagger \epsilon$. Furthermore, we can always rotate the spatial components of L in such a way that only the first component is non-vanishing. This then implies that

$$g(L, L) = L_0^2 - L_1^2. \quad (71)$$

$L_1 = \epsilon^\dagger \gamma_{01} \epsilon$ and if we combine this with $\gamma_{01}^\dagger = \gamma_{01}$, $\gamma_{01}^2 = 1$ and $\text{Tr}(\gamma_{01}) = 0$ we can use a $\text{SO}(\lfloor n/2 \rfloor)$ rotation to write $\gamma_{01} = \text{diag}([+]^{\lfloor n/2 \rfloor}, [-]^{\lfloor n/2 \rfloor})$. Decomposing the spinor w.r.t. the structure of γ_{01} as $\epsilon^t = (v, w)$, where v and w are vectors in $\mathbb{C}^{\lfloor n/2 \rfloor}$, we see that

$$L_0 = |v|^2 + |w|^2, \quad L_1 = |v|^2 - |w|^2 \quad \longrightarrow \quad g(L, L) = 4|v|^2|w|^2, \quad (72)$$

which implies the positive-definiteness of $|L|^2$.

In the derivation of the spinorial rule eq. (1) we have not made any particular assumption about the nature of the spinor ϵ which has been taken to be a (general) plain Dirac spinor. In the construction of the bilinears, however, it is wise to impose a bit more structure on ϵ ; this naturally leads one to investigate the compatibility of eq. (1) with the conditions for the existence of a Weyl, Majorana, Majorana-Weyl *etc.* spinor, a question that is answered affirmatively.

C Kundt metrics

A Kundt metric is a type of wave-like metric that allows for an expansion, shear and twist-free geodesic null-vector [28] and were first studied in the arbitrary- n case in refs. [29] and [30]. The line-element can always be taken to be

$$ds^2 = \hat{E}^+ \otimes \hat{E}^- + \hat{E}^- \otimes \hat{E}^+ - \hat{E}^x \otimes \hat{E}^x \quad (73)$$

where generically we introduce the light-cone-frame by⁸

$$\begin{cases} E^+ &= du \\ E^- &= dv + Hdu + S_m dy^m \\ E^x &= e_m^x dy^m \end{cases}, \quad \begin{cases} \theta_+ &= \partial_u - H\partial_v, \\ \theta_- &= \partial_v, \\ \theta_x &= e_x^m [\partial_m - S_m \partial_v] \end{cases}, \quad (74)$$

⁸ In order not to confuse the reader we define the directional derivatives θ_a to be the duals of the frame 1-forms E^a , *i.e.* we have $E^a(\theta_b) = \delta^a_b$. We shall reserve the notation ∂_x for the directional derivative on the base space, namely $\partial_x \equiv e_x^m \partial_m$.

where the Vielbein on the base space e_i^x is independent of v ; the only v -dependence resides in H and $\hat{S} \equiv S_m dy^m$.

This is the kind of metric that appeared in the characterisations of the null cases above, eqs. (37) and (55), where we defined the correspondence between the $(n-2)$ -bein and the base space metric as $h_{mn} \equiv e_m^i e_n^i$.

Defining the spin-connection $\omega^a_b \equiv E^c \omega_c^a{}_b$ by means of $dE^a = \omega^a_b \wedge E^b$ and imposing it to be metric compatible $\omega_{(ab)} = 0$, leads to

$$\omega_{+-} = -\theta_- H E^+ - \frac{1}{2} \theta_- S_x E^x, \quad (75)$$

$$\omega_{+x} = -(\theta_x H - e_x^m \theta_+ S_m) E^+ + \frac{1}{2} \theta_- S_x E^- - \left[T_{yx} + e_{(y}^m \theta_+ e_{x)m} \right] E^y, \quad (76)$$

$$\omega_{-x} = \frac{1}{2} \theta_- S_x E^+, \quad (77)$$

$$\omega_{xy} = -\lambda_{zy} E^z - \left[T_{xy} - e_{[x}^m \theta_+ e_{y]m} \right] E^+, \quad (78)$$

where we defined $dE^x = \lambda^x{}_y E^y$ and also defined $\lambda_{zy} = \delta_{zx} \lambda^x{}_y$, whereas $\omega_{xy} = \eta_{xz} \omega^z{}_y$ so that the sign difference is paramount.⁹ Furthermore, we defined

$$T_{xy} \equiv e_{[x}^m \theta_{y]} S_m \quad \text{which for } n=6 \text{ reads: } T_{ij} = v F_{ij} - \frac{1}{2} [D\varpi]_{ij}. \quad (79)$$

If we impose that the only u -dependency resides in H , the non-vanishing components of the Ricci tensor become

$$R_{++} = -\nabla_x^{(\lambda)} \partial_x H + \theta_- H \nabla_x^{(\lambda)} S_x - H \nabla_x^{(\lambda)} \theta_- S_x + 2S_x \partial_x \theta_- H - \theta_- S_x \partial_x H - S_x S_x \theta_-^2 H, \quad (80)$$

$$R_{+-} = -\theta_-^2 H - \frac{1}{2} \theta_- S_x \theta_- S_x + \frac{1}{2} \nabla_x^{(\lambda)} \theta_- S_x, \quad (81)$$

$$R_{+x} = -\theta_x \theta_- H - \nabla_y^{(\lambda)} T_{xy} + S_y \theta_- T_{xy} + T_{xy} \theta_- S_y, \quad (82)$$

$$R_{xy} = R(\lambda)_{xy} - \nabla_{(x}^{(\lambda)} \theta_- S_{|y)} + \frac{1}{2} \theta_- S_x \theta_- S_y, \quad (83)$$

The Ricci scalar is then given by

$$R = -2\theta_-^2 H - \frac{3}{2} \theta_- S_x \theta_- S_x + 2\nabla_x^{(\lambda)} \theta_- S_x - R(\lambda). \quad (84)$$

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⁹Observe that a similar condition holds for defining $e_{mx} = e_m^x$.

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